Some Results on Lagrangians of Hypergraphs

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Abstract

In 1965, Motzkin and Straus [5] provided a new proof of Turán's theorem based on a continuous characterization of the clique number of a graph using the Lagrangian of a graph. This new proof aroused interests in the study of Lagrangians of r-uniform graphs. The Lagrangian of a hypergraph has been a useful tool in hypergraph extremal problems. Sidorenko and Frankl-Füredi applied Lagrangians of hypergraphs in finding Turán densities of hypergraphs. Frankl and Rödl applied it in disproving Erdös' jumping constant conjecture. In most applications, we need an upper bound for the Lagrangian of a hypergraph. Frankl and Füredi conjectured that the r-uniform graph with m edges formed by taking the first m sets in the colex ordering of $\mathbb{N}^{(r)}$ has the largest Lagrangian of all r-uniform graphs with m edges. Talbot in [14] provided some evidence for Frankl and Füredi's conjecture. In this paper, we prove that the r-uniform graph with m edges formed by taking the first m sets in the colex ordering of $\mathbb{N}^{(r)}$ has the largest Lagrangian of all r-uniform graphs on t vertices with m edges when $m = \binom{t}{r} - 3$ or $\binom{t}{r} - 4$. As an implication, we also prove that Frankl and Füredi's conjecture holds for 3-uniform graphs with $m = \binom{t}{3} - 3$ or $\binom{t}{3} - 4$ edges.

Key Words: Lagrangians of r-uniform graphs, Extremal problems in hypergraphs

1 Introduction

In 1941, Turán [15] provided an answer to the following question: What is the maximum number of edges in a graph on n vertices without containing a complete subgraph of order t, for a given t? This is the well-known Turán theorem. Later, in another classical paper, Motzkin and Straus [5] provided a new proof of Turán's theorem based on a continuous characterization of the clique number of a graph using Lagrangians of graphs. This new proof aroused interests in the study of Lagrangians of r-graphs. The Lagrangian of a hypergraph has been a useful tool in hypergraph extremal problems. For example, Sidorenko [11] and Frankl-Füredi [1] applied Lagrangians of hypergraphs in finding Turán densities of hypergraphs. Frankl and Rödl [2] applied it in disproving Erdös' jumping constant conjecture. More applications of Lagrangians can be found in [12], [3] and [6]. In most applications, we need an upper bound for the Lagrangian of a hypergraph. In the course of estimating Turán densities of some

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hypergraphs, Frankl and Füredi [1] asked the following question: Given $r \geq 3$ and $m \in \mathbb{N}$, how large can the Lagrangian of an r-graph with m edges be? Before stating their conjecture on this problem, we give some definitions and notation.

For a set V and a positive integer r we denote by $V^{(r)}$ the family of all r-subsets of V. An r-uniform graph or r-graph G consists of a set $V(G)$ of vertices and a set $E(G) \subseteq V(G)^{(r)}$ of edges. An edge $e = \{a_1, a_2, \ldots, a_r\}$ will be simply denoted by $a_1a_2\ldots a_r$. An r-graph H is a *subgraph* of an r-graph G, denoted by $H \subseteq G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let $\mathbb{N}$ be the set of all positive integers. For any $n \in \mathbb{N}$, we denote the set $\{1, 2, 3, \ldots, n\}$ by $[n]$. Let $K_t^{(r)}$ denote the complete r-graph on t vertices, that is the r-graph on t vertices containing all possible edges. A complete r-graph on t vertices is also called a clique with order t. We also let $[n]^{(r)}$ represent the complete r-uniform graph on the vertex set $[n]$. When $r = 2$, an r-uniform graph is a simple graph. When $r \geq 3$, an r-graph is often called a hypergraph.

For an r-graph $G = (V, E)$ we denote the $(r-1)$-neighborhood of a vertex $i \in V$ by $E_i = \{A \in V^{(r-1)} : A \cup \{i\} \in E\}$. Similarly, we will denote the $(r-2)$-neighborhood of a pair of vertices $i, j \in V$ by $E_{ij} = \{B \in V^{(r-2)} : B \cup \{i, j\} \in E\}$. We denote the complement of E_i by $E_i^c = \{A \in V^{(r-1)} : A \cup \{i\} \in V^{(r)} \backslash E\}$. Also, we denote the complement of E_{ij} by $E_{ij}^c = \{B \in V^{(r-2)} : B \cup \{i, j\} \in V^{(r)} \backslash E\}$. Denote

$$E_{i \backslash j} = E_i \cap E_j^c.$$

Definition 1.1 *For an r-uniform graph G with the vertex set $[n]$, edge set $E(G)$ and a vector $\vec{x} = (x_1, \ldots, x_n) \in R^n$, define*

$$\lambda(G, \vec{x}) = \sum_{i_1 i_2 \cdots i_r \in E(G)} x_{i_1} x_{i_2} \ldots x_{i_r}.$$

Let $S = \{\vec{x} = (x_1, x_2, \ldots, x_n) : \sum_{i=1}^n x_i = 1, x_i \geq 0 \text{ for } i = 1, 2, \ldots, n\}$. The Lagrangian of G, denote by $\lambda(G)$, is defined as

$$\lambda(G) = \max\{\lambda(G, \vec{x}) : \vec{x} \in S\}.$$

We call $\vec{x} = (x_1, x_2, \ldots, x_n) \in R^n$ a legal weighting for G if $\vec{x} \in S$. A vector $\vec{y} \in S$ is called an optimal weighting *for G if $\lambda(G, \vec{y}) = \lambda(G)$.*

The following fact is easily implied by the definition of the Lagrangian.

Fact 1.1 *Let G_1, G_2 be r-uniform graphs and $G_1 \subseteq G_2$. Then $\lambda(G_1) \leq \lambda(G_2)$.*

In [5], Motzkin and Straus provided the following simple expression for the Lagrangian of a 2-graph.

Theorem 1.2 *(Motzkin and Straus [5]) If G is a 2-graph in which a largest clique has order t then $\lambda(G) = \lambda(K_t^{(2)}) = \frac{1}{2}(1 - \frac{1}{t})$.*

An attempt to generalize the Motzkin-Straus theorem to hypergraphs is due to Sós and Straus [13]. Recently, in [9] and [10] Rota Buló and Pelillo generalized the Motzkin and Straus' result to r-graphs in some way using a continuous characterization of maximal cliques. Determining the Lagrangian of a general r-graph is non-trivial when $r \geq 3$. Indeed the obvious generalization of Motzkin and Straus' result is false because there are many examples of r-graphs that do not achieve their Lagrangian on any proper subhypergraph.

For distinct $A, B \in \mathbb{N}^{(r)}$ we say that A is less than B in the *colex ordering* if $max(A \triangle B) \in B$, where $A \triangle B = (A \backslash B) \cup (B \backslash A)$. For example we have $246 < 156$ in $\mathbb{N}^{(3)}$ since $max(\{2,4,6\} \triangle \{1,5,6\}) \in \{1,5,6\}$. In colex ordering, $123 < 124 < 134 < 234 < 125 < 135 < 235 < 145 < 245 < 345 < 126 < 136 < 236 < 146 < 246 < 346 < 156 < 256 < 356 < 456 < 127 < \cdots$. The following conjecture of Frankl and Füredi (if it is true) proposes a solution to the question mentioned at the beginning.

Conjecture 1.3 *(Frankl and Füredi [1]) The r-graph with m edges formed by taking the first m sets in the colex ordering of $\mathbb{N}^{(r)}$ has the largest Lagrangian of all r-graphs with m edges. In particular, the r-graph with $\binom{t}{r}$ edges and the largest Lagrangian is $[t]^{(r)}$.*

This conjecture is true when $r = 2$ by Theorem 1.2. For the case $r = 3$, Talbot in [14] proved the following.

Theorem 1.4 *(Talbot [14]) Let m and t be integers satisfying*

$$\binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2} - (t-1).$$

Then Conjecture 1.3 is true for $r = 3$ and this value of m. Conjecture 1.3 is also true for $r = 3$ and $m = \binom{t}{3} - 1$ or $m = \binom{t}{3} - 2$.

The truth of Frankl and Füredi's conjecture is not known in general for $r \geq 4$. Even in the case $r = 3$, Theorem 1.4 does not cover the case when $\binom{t-1}{3} + \binom{t-2}{2} - (t-2) \leq m \leq \binom{t}{3} - 3$ in this conjecture. In [4], He, Peng, and Zhao verified Frankl and Füredi's conjecture for small values m when $r = 3$. In [7], Peng and Zhao generalized Theorem 1.4 further. Talbot in [14] proved some result as evidence of the truth of Conjecture 1.3 for r-graphs supported on $t+1$ vertices with $m = \binom{t}{r}$ edges. Also, in [8] Peng, Tang, and Zhao provided more results for Conjecture 1.3 when a given r-graph on t vertices satisfies some conditions.

Let $C_{r,m}$ denote the r-graph with m edges formed by taking the first m sets in the colex ordering of $\mathbb{N}^{(r)}$.

Lemma 1.5 *[14] For integers m, t, and r satisfying $\binom{t-1}{r} \leq m \leq \binom{t-1}{r} + \binom{t-2}{r-1}$, we have $\lambda(C_{r,m}) = \lambda([t-1]^{(r)})$.*

Definition 1.2 *An r-graph $G = (V, E)$ on the vertex set $[n]$ is left-compressed if $j_1 j_2 \cdots j_r \in E$ implies $i_1 i_2 \cdots i_r \in E$ provided $i_p \leq j_p$ for every $p, 1 \leq p \leq r$. Equivalently, G is left-compressed if $E_{j \backslash i} = \emptyset$ for any $1 \leq i < j \leq n$.*

Denote

$$\lambda_m^r = \max\{\lambda(G) : G \text{ is an } r-\text{graph with } m \text{ edges}\}.$$

The following lemma implies that we only need to consider left-compressed r-graphs when Conjecture 1.3 is discussed.

Lemma 1.6 *[14] Let m, t be positive integers satisfying $m \leq \binom{t}{r} - 1$, then there exists a left-compressed r-graph G with m edges such that $\lambda(G) = \lambda_m^r$.*

In this paper, we show that

Theorem 1.7 *Let m and t be positive integers satisfying $m = \binom{t}{r} - 3$. Then the r-graph with m edges formed by taking the first m sets in the colex ordering of $\mathbb{N}^{(r)}$ has the largest Lagrangian of all r-graphs with m edges and t vertices.*

Combining with a result in [14] (Lemma 3.3), we show the following corollary for $r = 3$.

Corollary 1.8 *Let m and t be positive integers satisfying $m = \binom{t}{3} - 3$. Then Conjecture 1.3 is true for $r = 3$ and this value of m.*

Theorem 1.9 *Let m and t be positive integers satisfying $m = \binom{t}{r} - 4$. Then the r-graph with m edges formed by taking the first m sets in the colex ordering of $\mathbb{N}^{(r)}$ has the largest Lagrangian of all r-graphs with m edges and t vertices.*

Combining with a result in [14] (Lemma 3.3), we show the following corollary for $r = 3$.

Corollary 1.10 *Let m and t be positive integers satisfying $m = \binom{t}{3} - 4$. Then Conjecture 1.3 is true for $r = 3$ and this value of m.*

The proof of Theorem 1.7 and Corollary 1.8 will be given in Section 3 and the proof of Theorem 1.9 and Corollary 1.10 will be given in Section 4. Next, we state some useful results.

2 Useful Results

We will impose one additional condition on any optimal weighting $\vec{x} = (x_1, x_2, \dots, x_n)$ for an r-graph G:

$$\begin{aligned} &|\{i : x_i > 0\}| \text{ is minimal, i.e. if } \vec{y} \text{ is a legal weighting for } G \text{ satisfying} \\ &|\{i : y_i > 0\}| < |\{i : x_i > 0\}|, \text{ then } \lambda(G, \vec{y}) < \lambda(G). \end{aligned} \tag{1}$$

When the theory of Lagrange multipliers is applied to find the optimum of $\lambda(G)$, subject to $\sum_{i=1}^{n} x_i = 1$, notice that $\lambda(E_i, \vec{x})$ corresponds to the partial derivative of $\lambda(G, \vec{x})$ with respect to x_i. The following lemma gives some necessary condition of an optimal weighting of G.

Lemma 2.1 *(Frankl and Rödl [2]) Let $G = (V, E)$ be an r-graph on the vertex set $[n]$ and $\vec{x} = (x_1, x_2, \dots, x_n)$ be an optimal weighting for G with k ($\leq n$) non-zero weights x_1, x_2, $\cdots$, x_k satisfying condition (1). Then for every $\{i, j\} \in [k]^{(2)}$, (a) $\lambda(E_i, \vec{x}) = \lambda(E_j, \vec{x}) = r\lambda(G)$, (b) there is an edge in E containing both i and j.*

Remark 2.2 *(a) In Lemma 2.1, part(a) implies that*

$$x_j \lambda(E_{ij}, \vec{x}) + \lambda(E_{i\setminus j}, \vec{x}) = x_i \lambda(E_{ij}, \vec{x}) + \lambda(E_{j\setminus i}, \vec{x}).$$

In particular, if G is left-compressed, then

$$(x_i - x_j)\lambda(E_{ij}, \vec{x}) = \lambda(E_{i\setminus j}, \vec{x})$$

for any i, j satisfying $1 \leq i < j \leq k$ since $E_{j\setminus i} = \emptyset$.

(b) If G is left-compressed, then for any i, j satisfying $1 \leq i < j \leq k$,

$$x_i - x_j = \frac{\lambda(E_{i\setminus j}, \vec{x})}{\lambda(E_{ij}, \vec{x})} \tag{2}$$

holds. If G is left-compressed and $E_{i\setminus j} = \emptyset$ for i, j satisfying $1 \leq i < j \leq k$, then $x_i = x_j$.

(c) By (2), if G is left-compressed, then an optimal weighting $\vec{x} = (x_1, x_2, \dots, x_n)$ for G must satisfy

$$x_1 \geq x_2 \geq \ldots \geq x_n \geq 0. \tag{3}$$

3 Proofs of Theorem 1.7 and Corollary 1.8

Denote

$$\lambda^r_{(m,t)} = \max\{\lambda(G) : G \text{ is an } r\text{−graph with } m \text{ edges and } t \text{ vertices}\}.$$

An r-tuple $i_1 i_2 \cdots i_r$ is called a *descendant* of an r-tuple $j_1 j_2 \cdots j_r$ if $i_s \leq j_s$ for each $1 \leq s \leq r$, and $i_1 + i_2 + \cdots + i_r < j_1 + j_2 + \cdots + j_r$. In this case, the r-tuple $j_1 j_2 \cdots j_r$ is called an *ancestor* of $i_1 i_2 \cdots i_r$. The r-tuple $i_1 i_2 \cdots i_r$ is called a *direct descendant* of $j_1 j_2 \cdots j_r$ if $i_1 i_2 \cdots i_r$ is a descendant of $j_1 j_2 \cdots j_r$ and $j_1 + j_2 + \cdots + j_r = i_1 + i_2 + \cdots + i_r + 1$. We say that $i_1 i_2 \cdots i_r$ has lower hierarchy than $j_1 j_2 \cdots j_r$ if $i_1 i_2 \cdots i_r$ is a descendant of $j_1 j_2 \cdots j_r$. This is a partial order on the set of all r-tuples. Figure 1 is a Hessian diagram on all r-tuples on $[t]$. In this diagram, $i_1 i_2 \cdots i_r$ and $j_1 j_2 \cdots j_r$ are connected by an edge if $i_1 i_2 \cdots i_r$ is a direct descendant of $j_1 j_2 \cdots j_r$.

Remark 3.1 *An r-graph G is left-compressed if and only if all descendants of an edge of G are edges of G. Equivalently, if an r-tuple is not an edge of G, then none of its ancestors will be an edge of G.*

Lemma 3.2 *There exists a left-compressed r-graph G on the vertex set $[t]$ with m edges such that $\lambda(G) = \lambda^r_{(m,t)}$.*

Proof. Let $G' = (V, E)$ be an r-graph on the vertex set $[t]$ with m edges such that $\lambda(G') = \lambda^r_{(m,t)}$. We call such an r-graph G' an extremal r-graph for m and t. Let $\vec{x} = (x_1, x_2, \dots, x_t)$ be an optimal weighting of G'. We can assume that $x_i \geq x_j$ when $i < j$ since otherwise we can just relabel the vertices of G' and obtain another extremal r-graph for m and t with an optimal weighting $\vec{x} = (x_1, x_2, \dots, x_t)$ satisfying $x_i \geq x_j$ when $i < j$. If G' is not left-compressed, then there is an edge such that at least one of its descendants is not an edge. Replace all those edges by its available descendants with the lowest hierarchy, then we get a left-compressed r-graph G on the vertex set $[t]$ with m edges and $\lambda(G, \vec{x}) \geq \lambda(G')$. Therefore, G is a left-compressed extremal r-graph for m and t. ■

Proof of Theorem 1.7. By Lemma 3.2, we only need to consider left-compressed r-graphs on vertex set $[t]$ with $m = \binom{t}{r} - 3$ edges. Every left-compressed r-graph on $[t]$ with $m = \binom{t}{r} - 3$ edges can be obtained by removing three r-tuples from $[t]^{(r)}$ such that if an r-tuple is removed then all its ancestors should be removed by Remark 3.1. In view of Figure 1, these three r-tuples to be removed are either

$$\{(t-r+1)(t-r+2)\dots(t-1)t, (t-r)(t-r+2)\dots(t-1)t, (t-r)(t-r+1)(t-r+3)\dots(t-1)t\}$$

or

$$\{(t-r+1)(t-r+2)\dots(t-1)t, (t-r)(t-r+2)\dots(t-1)t, (t-r-1)(t-r+2)\dots(t-1)t\}.$$

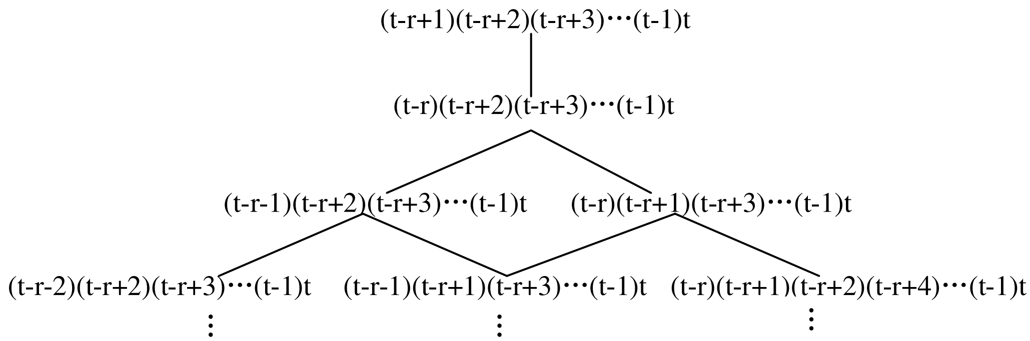

Figure 1

Therefore, there are only two different left-compressed r-graphs with $m = \binom{t}{r} - 3$ edges on $[t]$. They are

$$G_1 = ([t], E) \text{ with the edge set}$$

$$E = [t]^{(r)} \setminus \{(t-r+1)(t-r+2)\dots(t-1)t, (t-r)(t-r+2)\dots(t-1)t, (t-r)(t-r+1)(t-r+3)\dots(t-1)t\},$$

and

$$G_2 = ([t], E') \text{ with the edge set}$$

$$E' = [t]^{(r)} \setminus \{(t-r+1)(t-r+2)\dots(t-1)t, (t-r)(t-r+2)\dots(t-1)t, (t-r-1)(t-r+2)\dots(t-1)t\}.$$

Clearly, G_2 is formed by taking the first m sets in the colex ordering of $\mathbb{N}^{(r)}$. So in order to prove Theorem 1.7, we only need to prove $\lambda(G_1) \le \lambda(G_2)$.

Let $\vec{x} = (x_1, x_2, \dots, x_t)$ be an optimal weighting for G_1 satisfying $x_1 \ge x_2 \ge \dots \ge x_t \ge 0$. First note that $x_t > 0$. If $x_t = 0$, then $\lambda(G_1) = \lambda([t-1]^{(r)})$. However, if we take a legal weighting $\vec{x} = (x_1, \dots, x_t)$, where $x_1 = x_2 = \dots = x_{t-2} = \frac{1}{t-1}$ and $x_{t-1} = x_t = \frac{1}{2(t-1)}$, then $\lambda(G_1, \vec{x}) > \lambda([t-1]^{(r)})$. This contradiction implies that $x_t > 0$. Since G_1 is left-compressed and $E_{i\setminus j} = \emptyset$ for i, j satisfying $1 \le i < j \le t-r-1$, or $t-r \le i < j \le t-r+2$ or $t-r+3 \le i < j \le t$, by Remark 2.2(b), we have $x_1 = x_2 = \dots = x_{t-r-1} = a$, $x_{t-r} = x_{t-r+1} = x_{t-r+2} = b$, and $x_{t-r+3} = x_{t-r+4} = \dots = x_t = c$.

Consider a weighting for G_2: $\vec{y} = (y_1, y_2, \dots, y_t)$ given by $y_i = x_i$ for $i \ne t-r-1$, $i \ne t-r$ and $y_{t-r-1} = x_{t-r-1} - \delta$, $y_{t-r} = x_{t-r} + \delta$. Then

$$\begin{aligned} \lambda(G_2, \vec{y}) - \lambda(G_2, \vec{x}) &= \delta[\lambda(E'_{t-r}, \vec{x}) - \lambda(E'_{t-r-1}, \vec{x})] - \delta^2 \lambda(E'_{(t-r-1)(t-r)}, \vec{x}) \\ &= \delta(x_{t-r-1} - x_{t-r})\lambda(E'_{(t-r-1)(t-r)}, \vec{x}) - \delta^2 \lambda(E'_{(t-r-1)(t-r)}, \vec{x}) \\ &= \delta(a-b)\lambda(E'_{(t-r-1)(t-r)}, \vec{x}) - \delta^2 \lambda(E'_{(t-r-1)(t-r)}, \vec{x}). \qquad (4) \end{aligned}$$

Let $\delta = \frac{a-b}{2}$, then $y_{t-r-1} = a - \frac{a-b}{2} = \frac{a+b}{2} > 0$, and $y_{t-r} = b + \frac{a-b}{2} = \frac{a+b}{2}$. Hence $\vec{y} = (y_1, y_2, \dots, y_t)$ is also a legal weighting for G_2, and

$$\lambda(G_2, \vec{y}) - \lambda(G_2, \vec{x}) = \frac{(a-b)^2}{4} \lambda(E'_{(t-r-1)(t-r)}, \vec{x}).$$

So

$$\lambda(G_2, \vec{y}) - \lambda(G_1, \vec{x}) = \frac{(a-b)^2}{4} \lambda(E'_{(t-r-1)(t-r)}, \vec{x}) + \lambda(G_2, \vec{x}) - \lambda(G_1, \vec{x})$$

$$
\begin{aligned}
&= \frac{(a-b)^2}{4}\lambda(E'_{(t-r-1)(t-r)},\vec{x}) - (a-b)x_{t-r+2}x_{t-r+3}\dots x_{t-1}x_t \\
&= \frac{(a-b)}{4}[(a-b)\lambda(E'_{(t-r-1)(t-r)},\vec{x}) - 4x_{t-r+2}x_{t-r+3}\dots x_{t-1}x_t]. \qquad (5)
\end{aligned}
$$

By Remark 2.2(b), we have

$$
a-b = \frac{\lambda(E_{(t-r-1)\setminus(t-r)})}{\lambda(E_{(t-r-1)(t-r)})} = \frac{(x_{t-r+1}+x_{t-r+2})x_{t-r+3}\dots x_{t-1}x_t}{\lambda(E_{(t-r-1)(t-r)},\vec{x})} = \frac{2x_{t-r+2}x_{t-r+3}\dots x_{t-1}x_t}{\lambda(E'_{(t-r-1)(t-r)},\vec{x})} \qquad (6)
$$

since $x_{t-r+1} = x_{t-r+2}$ and $\lambda(E_{(t-r-1)(t-r)},\vec{x}) = \lambda(E'_{(t-r-1)(t-r)},\vec{x})$. So

$$
\lambda(G_2,\vec{y}) - \lambda(G_1,\vec{x}) = -\frac{x_{t-r+2}x_{t-r+3}\dots x_{t-1}x_t}{\lambda(E'_{(t-r-1)(t-r)},\vec{x})}x_{t-r+2}x_{t-r+3}\dots x_{t-1}x_t. \qquad (7)
$$

Consider a new weighting for G_2: $\vec{z} = (z_1, z_2, \dots, z_t)$ given by $z_i = x_i$ for $i \neq t-r+1, i \neq t-r+2$ and $z_{t-r+1} = y_{t-r+1} + \eta$, $z_{t-r+2} = y_{t-r+2} - \eta$. Then

$$
\begin{aligned}
\lambda(G_2,\vec{z}) - \lambda(G_2,\vec{y}) &= \eta[\lambda(E'_{t-r+1},\vec{y}) - \lambda(E'_{t-r+2},\vec{y})] - \eta^2\lambda(E'_{(t-r+1)(t-r+2)},\vec{y}) \\
&= \eta[y_{t-r}y_{t-r+3}\dots y_{t-1}y_t + y_{t-r-1}y_{t-r+3}\dots y_{t-1}y_t) \\
&\quad -(y_{t-r+1} - y_{t-r+2})\lambda(E'_{(t-r+1)(t-r+2)},\vec{y})] - \eta^2\lambda(E'_{(t-r+1)(t-r+2)},\vec{y}). \qquad (8)
\end{aligned}
$$

Note that $y_{t-r+1} = y_{t-r+2}$, we have

$$
\begin{aligned}
\lambda(G_2,\vec{z}) - \lambda(G_2,\vec{y}) &= \eta(y_{t-r}y_{t-r+3}\dots y_{t-1}y_t + y_{t-r-1}y_{t-r+3}\dots y_{t-1}y_t) - \\
&\quad -\eta^2\lambda(E'_{(t-r+1)(t-r+2)},\vec{y}). \qquad (9)
\end{aligned}
$$

Let

$$
\eta = \frac{(y_{t-r-1}+y_{t-r})y_{t-r+3}\dots y_{t-1}y_t}{2\lambda(E'_{(t-r+1)(t-r+2)},\vec{y})}.
$$

Since

$$
12\dots(r-2) \in E'_{(t-r+1)(t-r+2)},
$$

then

$$
\eta = \frac{(y_{t-r-1}+y_{t-r})y_{t-r+3}\dots y_{t-1}y_t}{2\lambda(E'_{(t-r+1)(t-r+2)},\vec{y})} \leq \frac{(y_{t-r-1}+y_{t-r})y_{t-r+3}\dots y_{t-1}y_t}{2y_1y_2\dots y_{r-2}} \leq y_t.
$$

Hence $\vec{z} = (z_1, z_2, \dots, z_t)$ is also a legal weighting for G_2, and

$$
\begin{aligned}
\lambda(G_2,\vec{z}) - \lambda(G_2,\vec{y}) &= \frac{(y_{t-r-1}+y_{t-r})^2y^2_{t-r+3}\dots y^2_{t-1}y^2_t}{2\lambda(E'_{(t-r+1)(t-r+2)},\vec{y})} - \\
&\quad -[\frac{(y_{t-r-1}+y_{t-r})y_{t-r+3}\dots y_{t-1}y_t}{2\lambda(E'_{(t-r+1)(t-r+2)},\vec{y})}]^2\lambda(E'_{(t-r+1)(t-r+2)},\vec{y}) \\
&= \frac{(y_{t-r-1}+y_{t-r})^2y^2_{t-r+3}\dots y^2_{t-1}y^2_t}{4\lambda(E'_{(t-r+1)(t-r+2)},\vec{y})}. \qquad (10)
\end{aligned}
$$

Using (7) and (10), we have

$$
\begin{aligned}
\lambda(G_2,\vec{z}) - \lambda(G_1,\vec{x}) &= \frac{(y_{t-r-1}+y_{t-r})^2y^2_{t-r+3}\dots y^2_{t-1}y^2_t}{4\lambda(E'_{(t-r+1)(t-r+2)},\vec{y})} \\
&\quad -\frac{x^2_{t-r+2}x^2_{t-r+3}\dots x^2_{t-1}x^2_t}{\lambda(E'_{(t-r-1)(t-r)},\vec{x})}. \qquad (11)
\end{aligned}
$$

Note that $y_{t-r-1}+y_{t-r}=x_{t-r-1}+x_{t-r}=a+b, y_{t-r+3}=x_{t-r+3},\dots,y_t=x_t=c$; and

$$\lambda(E'_{(t-r-1)(t-r)},\vec{x})=\lambda(E'_{(t-r-1)(t-r)},\vec{y}).$$

So

$$\begin{aligned}\lambda(G_2,\vec{z})-\lambda(G_1,\vec{x}) &= \frac{(a+b)^2c^{2(r-2)}}{4\lambda(E'_{(t-r+1)(t-r+2)},\vec{y})}-\frac{b^2c^{2(r-2)}}{\lambda(E'_{(t-r-1)(t-r)},\vec{y})}\\ &\geq \frac{b^2c^{2(r-2)}}{\lambda(E'_{(t-r+1)(t-r+2)},\vec{y})}-\frac{b^2c^{2(r-2)}}{\lambda(E'_{(t-r-1)(t-r)},\vec{y})}.\end{aligned}$$

Since G_2 is left-compressed, we have

$$\lambda(E'_{(t-r+1)(t-r+2)},\vec{y})\leq\lambda(E'_{(t-r-1)(t-r)},\vec{y}).$$

Hence

$$\lambda(G_2)\geq\lambda(G_2,\vec{z})\geq\lambda(G_1,\vec{x})=\lambda(G_1).$$

This proves the theorem. ∎

Proof of Corollary 1.8. Let m and t be integers satisfying $\binom{t-1}{3}\leq m\leq\binom{t}{3}-1$. Let $G=(V,E)$ be a 3-graph with m edges such that $\lambda(G)=\lambda_m^3$. Applying Lemma 1.6, we can assume that G is left-compressed. Let $\vec{x}=(x_1,x_2,\dots,x_n)$ be an optimal weighting for G satisfying $x_1\geq x_2\geq\dots\geq x_k>x_{k+1}=\dots=x_n=0$.

In [14], the following result is proved.

Lemma 3.3 *(Talbot [14])*

$$|E|\geq\binom{k-1}{3}+\binom{k-2}{2}-(k-2).$$

We claim that $k\leq t$. Otherwise $k\geq t+1$ and Lemma 3.3 implies that

$$\begin{aligned}m=|E| &\geq \binom{k-1}{3}+\binom{k-2}{2}-(k-2)\\ &\geq \binom{t}{3}+\binom{t-1}{2}-(t-1)\\ &\geq \binom{t}{3}\end{aligned}$$

which contradicts to the assumption that $m=\binom{t}{3}-3$. Hence $k\leq t$. Combining with Theorem 1.7, we see that the corollary follows. ∎

4 Proof of Theorem 1.9 and Corollary 1.10

Proof of Theorem 1.9. By Lemma 3.2, we only need to consider left-compressed r-graphs on $[t]$ with $m=\binom{t}{r}-4$ edges. Every left-compressed r-graph on $[t]$ with $m=\binom{t}{r}-4$ edges can be obtained by removing four r-tuples from $[t]^{(r)}$ such that if an r-tuple is removed then all its ancestors should be removed. In view of Figure 1, these four r-tuples to be removed are

$$\{(t-r+1)(t-r+2)\dots t,(t-r)(t-r+2)\dots t,(t-r-1)(t-r+2)\dots t,(t-r-2)(t-r+2)\dots t\}$$

or

$$\{(t-r+1)(t-r+2)\dots t,(t-r)(t-r+2)\dots t,(t-r-1)(t-r+2)\dots t,(t-r)(t-r+1)(t-r+3)\dots t\}$$

or

$$\{(t-r+1)(t-r+2)\dots t,(t-r)(t-r+2)\dots t,(t-r)(t-r+1)(t-r+3)\dots t,(t-r)(t-r+1)(t-r+2)(t-r+4)\dots t\}.$$

Therefore, there are only three different left-compressed r-graphs with $m=\binom{t}{r}-4$ edges on $[t]$. They are

$G_1=([t],E)$ with the edge set

$$E=[t]^{(r)}\setminus\{(t-r+1)(t-r+2)\dots t,(t-r)(t-r+2)\dots t,(t-r-1)(t-r+2)\dots t,(t-r-2)(t-r+2)\dots t\},$$

$G_2=([t],E')$ with the edge set

$$E'=[t]^{(r)}\setminus\{(t-r+1)(t-r+2)\dots t,(t-r)(t-r+2)\dots t,(t-r-1)(t-r+2)\dots t,(t-r)(t-r+1)(t-r+3)\dots t\},$$

and

$G_3=([t],E'')$ with the edge set

$$E''=[t]^{(r)}\setminus\{(t-r+1)(t-r+2)\dots t,(t-r)(t-r+2)\dots t,(t-r)(t-r+1)(t-r+3)\dots t,$$
$$(t-r)(t-r+1)(t-r+2)(t-r+4)\dots t\}.$$

Clearly, G_1 is formed by taking the first m sets in the colex ordering of $\mathbb{N}^{(r)}$. So in order to prove Theorem 1.9, we only need to prove $\lambda(G_1)\geq\lambda(G_2)$ and $\lambda(G_1)\geq\lambda(G_3)$.

First, we show that $\lambda(G_2)\leq\lambda(G_1)$.

Let $\vec{x}=(x_1,x_2,\dots,x_t)$ be an optimal weighting for G_2 satisfying $x_1\geq x_2\geq\dots\geq x_t\geq 0$. Note that $x_t>0$. If $x_t=0$, then $\lambda(G_2)=\lambda([t-1]^{(r)})$. However, if we take a legal weighting $\vec{x}=(x_1,\dots,x_t)$, where $x_1=x_2=\dots=x_{t-2}=\frac{1}{t-1}$ and $x_{t-1}=x_t=\frac{1}{2(t-1)}$, then $\lambda(G_2,\vec{x})>\lambda([t-1]^{(r)})$. This contradiction implies that $x_t>0$. Since G_2 is left-compressed and $E'_{i\setminus j}=\emptyset$ for i,j satisfying $1\leq i<j\leq t-r-2$, or $t-r\leq i<j\leq t-r+1$ or $t-r+3\leq i<j\leq t$, by Remark 2.2(b), we have $x_1=x_2=\dots=x_{t-r-2}=a$, $x_{t-r-1}=b, x_{t-r}=x_{t-r+1}=c$, and $x_{t-r+2}=d$, $x_{t-r+3}=x_{t-r+4}=\dots=x_t=e$. Note that

$$\lambda(G_1,\vec{x})-\lambda(G_2,\vec{x})=c^2e^{r-2}-ade^{r-2}. \tag{12}$$

Consider a weighting for G_1: $\vec{y}=(y_1,y_2,\dots,y_t)$ given by $y_i=x_i$ for $i\neq t-r+1$, $i\neq t-r+2$ and $y_{t-r+1}=x_{t-r+1}+\delta$, $y_{t-r+2}=x_{t-r+2}-\delta$. Then

$$\begin{aligned}\lambda(G_1,\vec{y})-\lambda(G_1,\vec{x}) &= \delta[\lambda(E_{t-r+1},\vec{x})-\lambda(E_{t-r+2},\vec{x})]-\delta^2\lambda(E_{(t-r+1)(t-r+2)},\vec{x})\\ &= \delta[x_{t-r}x_{t-r+3}\dots x_t+x_{t-r-1}x_{t-r+3}\dots x_t+x_{t-r-2}x_{t-r+3}\dots x_t\\ &\quad -(x_{t-r+1}-x_{t-r+2})\lambda(E_{(t-r+1)(t-r+2)},\vec{x})]-\delta^2\lambda(E_{(t-r+1)(t-r+2)},\vec{x})\\ &= \delta[(a+b+c)e^{r-2}-(c-d)\lambda(E_{(t-r+1)(t-r+2)},\vec{x})]\\ &\quad -\delta^2\lambda(E_{(t-r+1)(t-r+2)},\vec{x}).\end{aligned} \tag{13}$$

Let

$$\delta=\frac{(a+b+c)e^{r-2}}{2\lambda(E_{(t-r+1)(t-r+2)},\vec{x})}-\frac{c-d}{2}=\frac{(a+b+c)e^{r-2}}{2\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})}-\frac{c-d}{2}.$$

By Remark 2.2(b), we have

$$c-d=\frac{\lambda(E'_{(t-r+1)\backslash(t-r+2)},\vec{x})}{\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})}=\frac{be^{r-2}}{\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})}.$$

So

$$\delta=\frac{(a+c)e^{r-2}}{2\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})}. \tag{14}$$

Clearly $\delta>0$. Since

$$1(t-r+4)\dots t\in E'_{(t-r+1)(t-r+2)},$$
$$1(t-r+3)(t-r+5)\dots t\in E'_{(t-r+1)(t-r+2)}$$

and in view of (14), $\delta<e$. So $\vec{y}=(y_1,y_2,\dots,y_t)$ is also a legal weighting for G_1 and

$$\lambda(G_1,\vec{y})-\lambda(G_1,\vec{x})=\frac{(a+c)^2e^{2(r-2)}}{4\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})} \tag{15}$$

since $\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})=\lambda(E_{(t-r+1)(t-r+2)},\vec{x})$.

Consider a new weighting for G_1: $\vec{z}=(z_1,z_2,\dots,z_t)$ given by $z_i=y_i$ for $i\neq t-r-2$, $i\neq t-r$ and $z_{t-r-2}=y_{t-r-2}-\eta$, $z_{t-r}=y_{t-r}+\eta$. Then

$$\begin{aligned}\lambda(G_1,\vec{z})-\lambda(G_1,\vec{y}) &= \eta[\lambda(E_{t-r},\vec{y})-\lambda(E_{t-r-2},\vec{y})]-\eta^2\lambda(E_{(t-r-2)(t-r)},\vec{y})\\ &= \eta(y_{t-r-2}-y_{t-r})\lambda(E_{(t-r-2)(t-r)},\vec{y})-\eta^2\lambda(E_{(t-r-2)(t-r)},\vec{y})\\ &= \eta(a-c)\lambda(E_{(t-r-2)(t-r)},\vec{y})-\eta^2\lambda(E_{(t-r-2)(t-r)},\vec{y}).\end{aligned} \tag{16}$$

Let $\eta=\frac{a-c}{2}$. Clearly, $\vec{z}=(z_1,z_2,\dots,z_t)$ is also a legal weighting for G_1, and

$$\lambda(G_1,\vec{z})-\lambda(G_1,\vec{y})=\frac{(a-c)^2}{4}\lambda(E_{(t-r-2)(t-r)},\vec{y}). \tag{17}$$

Adding (12), (15) and (17), we have

$$\begin{aligned}\lambda(G_1,\vec{z})-\lambda(G_2,\vec{x}) &= \frac{(a+c)^2e^{2(r-2)}}{4\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})}+\frac{(a-c)^2}{4}\lambda(E_{(t-r-2)(t-r)},\vec{y})+c^2e^{r-2}-ade^{r-2}\\ &\geq \frac{(a+c)^2e^{2(r-2)}}{4\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})}+\frac{(a-c)^2}{4}\lambda(E_{(t-r-2)(t-r)},\vec{y})-(a-c)de^{r-2}.\end{aligned} \tag{18}$$

By Remark 2.2(b), we have

$$a-c=\frac{\lambda(E'_{(t-r-2)\backslash(t-r)},\vec{x})}{\lambda(E'_{(t-r-2)(t-r)},\vec{x})}=\frac{(d+c)e^{r-2}}{\lambda(E'_{(t-r-2)(t-r)},\vec{x})}.$$

Hence

$$\begin{aligned}\lambda(G_1,\vec{z})-\lambda(G_2,\vec{x}) &\geq \frac{(a+c)^2e^{2(r-2)}}{4\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})}+\frac{(d+c)^2e^{2(r-2)}}{4(\lambda(E'_{(t-r-2)(t-r)},\vec{x}))^2}\lambda(E_{(t-r-2)(t-r)},\vec{y})\\ &\quad -\frac{(d+c)de^{2(r-2)}}{\lambda(E'_{(t-r-2)(t-r)},\vec{x})}\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{(a+c)^2e^{2(r-2)}}{4\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})}+\frac{(d+c)^2e^{2(r-2)}}{4(\lambda(E'_{(t-r-2)(t-r)},\vec{x}))^2}\lambda(E_{(t-r+1)(t-r+2)},\vec{y})\\
&\quad -\frac{(d+c)^2e^{2(r-2)}}{2\lambda(E'_{(t-r-2)(t-r)},\vec{x})}\\
&= \frac{(a+c)^2e^{2(r-2)}}{4\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})}+\frac{(d+c)^2e^{2(r-2)}}{4(\lambda(E'_{(t-r-2)(t-r)},\vec{x}))^2}\lambda(E_{(t-r+1)(t-r+2)},\vec{x})\\
&\quad -\frac{(d+c)^2e^{2(r-2)}}{2\lambda(E'_{(t-r-2)(t-r)},\vec{x})}\\
&= \frac{(a+c)^2e^{2(r-2)}}{4\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})}+\frac{(d+c)^2e^{2(r-2)}}{4(\lambda(E'_{(t-r-2)(t-r)},\vec{x}))^2}\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})\\
&\quad -\frac{(d+c)^2e^{2(r-2)}}{2\lambda(E'_{(t-r-2)(t-r)},\vec{x})}
\end{aligned} \tag{19}
$$

Observe that

$$
\begin{aligned}
&[\frac{(d+c)^2e^{2(r-2)}\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})}{4(\lambda(E'_{(t-r-2)(t-r)},\vec{x}))^2}-\frac{(d+c)^2e^{2(r-2)}}{2\lambda(E'_{(t-r-2)(t-r)},\vec{x})}]\\
&-[\frac{(d+c)^2e^{2(r-2)}\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})}{4(\lambda(E'_{(t-r+1)(t-r+2)},\vec{x}))^2}-\frac{(d+c)^2e^{2(r-2)}}{2\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})}]\\
= \; &(d+c)^2e^{2(r-2)}[\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})(\frac{1}{4(\lambda(E'_{(t-r-2)(t-r)},\vec{x}))^2}-\frac{1}{4(\lambda(E'_{(t-r+1)(t-r+2)},\vec{x}))^2})\\
&-(\frac{1}{2\lambda(E'_{(t-r-2)(t-r)},\vec{x})}-\frac{1}{2\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})})]\\
= \; &(d+c)^2e^{2(r-2)}(\frac{1}{2\lambda(E'_{(t-r-2)(t-r)},\vec{x})}-\frac{1}{2\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})})\\
&\times[\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})(\frac{1}{2\lambda(E'_{(t-r-2)(t-r)},\vec{x})}+\frac{1}{2\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})})-1]\\
\geq \; &0.
\end{aligned} \tag{20}
$$

The last inequality is true because of the following: since G is left-compressed, then $\lambda(E'_{(t-r+1)(t-r+2)},\vec{x}) \leq \lambda(E'_{(t-r-2)(t-r)},\vec{x})$. So $\frac{1}{2\lambda(E'_{(t-r-2)(t-r)},\vec{x})}-\frac{1}{2\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})} \leq 0$ and $\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})(\frac{1}{2\lambda(E'_{(t-r-2)(t-r)},\vec{x})}+\frac{1}{2\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})})-1 \leq 0$.

Combining (19) and (20), we have

$$
\begin{aligned}
\lambda(G_1,\vec{z})-\lambda(G_2,\vec{x}) &\geq \frac{(a+c)^2e^{2(r-2)}}{4\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})}+\frac{(d+c)^2e^{2(r-2)}}{4(\lambda(E'_{(t-r+1)(t-r+2)},\vec{x}))^2}\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})\\
&\quad -\frac{(d+c)^2e^{2(r-2)}}{2\lambda(E'_{(t-r+1)(t-r+2)},\vec{x})}\\
&\geq 0.
\end{aligned}
$$

Hence

$$\lambda(G_1) \geq \lambda(G_1,\vec{z}) \geq \lambda(G_2,\vec{x}) = \lambda(G_2).$$

Next we prove $\lambda(G_3) \le \lambda(G_1)$.

Let $\vec{x'} = (x'_1, x'_2, \dots, x'_t)$ be an optimal weighting for G_3 satisfying $x'_1 \ge x'_2 \ge \dots \ge x'_t \ge 0$. Note that $x'_t > 0$. If $x'_t = 0$, then $\lambda(G_3) = \lambda([t-1]^{(r)})$. However, if we take a legal weighting $\vec{x'} = (x'_1, \dots, x'_t)$, where $x'_1 = x'_2 = \dots = x'_{t-2} = \frac{1}{t-1}$ and $x'_{t-1} = x'_t = \frac{1}{2(t-1)}$, then $\lambda(G_3, \vec{x'}) > \lambda([t-1]^{(r)})$. This contradiction implies that $x'_t > 0$. Since G_3 is left-compressed and $E''_{i\setminus j} = \emptyset$ for i, j satisfying $1 \le i < j \le t-r-1$, or $t-r \le i < j \le t-r+3$ or $t-r+4 \le i < j \le t$, by Remark 2.2(b), we have $x'_1 = x'_2 = \dots = x'_{t-r-1} = a'$, $x'_{t-r} = x'_{t-r+1} = x'_{t-r+2} = x'_{t-r+3} = b'$, $x'_{t-r+4} = x'_{t-r+5} = \dots = x'_t = c'$. Note that

$$\lambda(G_1, \vec{x'}) - \lambda(G_3, \vec{x'}) = (2b'^3 - 2a'b'^2)c'^{r-3}. \tag{21}$$

By Remark 2.2(b), we have

$$a' - b' = \frac{\lambda(E''_{(t-r-1)\setminus(t-r)}, \vec{x})}{\lambda(E''_{(t-r-1)(t-r)}, \vec{x})} = \frac{3b'^2c'^{t-3}}{\lambda(E''_{(t-r-1)(t-r)}, \vec{x})} \le c' \le b'.$$

Therefore,

$$a' \le 2b'. \tag{22}$$

Consider a weighting for G_1: $\vec{y'} = (y'_1, y'_2, \dots, y'_t)$ given by $y'_i = x'_i$ for $i \ne t-r$, $i \ne t-r+3$ and $y'_{t-r} = x'_{t-r} + \frac{a'-b'}{3} = \frac{a'+2b'}{3}$, $y'_{t-r+3} = x'_{t-r+3} - \frac{a'-b'}{3} = \frac{4b'-a'}{3}$. Clearly, $\vec{y'} = (y'_1, y'_2, \dots, y'_t)$ is a legal weighting for G_1, and

$$\begin{aligned}
\lambda(G_1, \vec{y'}) - \lambda(G_1, \vec{x'}) &= \frac{a'-b'}{3}[\lambda(E_{t-r}, \vec{x'}) - \lambda(E_{t-r+3}, \vec{x'})] - (\frac{a'-b'}{3})^2\lambda(E_{(t-r)(t-r+3)}, \vec{x'}) \\
&= \frac{a'-b'}{3}[x'_{t-r+1}x'_{t-r+2}x'_{t-r+4}\dots x'_t + x'_{t-r-1}x'_{t-r+2}x'_{t-r+4}\dots x'_t + \\
&\quad +x'_{t-r-2}x'_{t-r+2}x'_{t-r+4}\dots x'_{t-r} + (x'_{t-r+3} - x'_{t-r})\lambda(E_{(t-r)(t-r+3)}, \vec{x'})] \\
&\quad -(\frac{a'-b'}{3})^2\lambda(E_{(t-r)(t-r+3}, \vec{x'}) \\
&= \frac{a'-b'}{3}(b'^2c^{r-3} + 2a'b'c^{r-3}) - \frac{(a'-b')^2}{9}\lambda(E_{(t-r)(t-r+3)}, \vec{x'}).
\end{aligned} \tag{23}$$

Consider a new weighting for G_1: $\vec{z'} = (z'_1, z'_2, \dots, z'_t)$ given by $z'_i = y'_i$ for $i \ne t-r+1, i \ne t-r+2$ and $z'_{t-r+1} = y'_{t-r+1} + \frac{a'-b'}{3} = \frac{a'+2b'}{3}$, $z'_{t-r+2} = y'_{t-r+2} - \frac{a'-b'}{3} = \frac{4b'-a'}{3}$. Clearly $\vec{z'} = (z'_1, z'_2, \dots, z'_t)$ is also a legal weighting for G_1, and

$$\begin{aligned}
\lambda(G_1, \vec{z'}) - \lambda(G_1, \vec{y'}) &= \frac{a'-b'}{3}[\lambda(E_{t-r+1}, \vec{y'}) - \lambda(E_{t-r+2)}, \vec{y'})] - (\frac{a'-b'}{3})^2\lambda(E_{(t-r+1)(t-r+2)}, \vec{y'}) \\
&= \frac{a'-b'}{3}[y'_{t-r}y'_{t-r+3}\cdots y'_t + y'_{t-r-1}y'_{t-r+3}\cdots y'_t + y'_{t-r-2}y'_{t-r+3}\cdots y'_t \\
&\quad -(y'_{t-r+1} - y'_{t-r+2})\lambda(E_{(t-r+1)(t-r+2)}, \vec{y'})] - (\frac{a'-b'}{3})^2\lambda(E_{(t-r+1)(t-r+2)}, \vec{y'}) \\
&= \frac{a'-b'}{3}c'^{r-3}(\frac{a'+2b'}{3}\frac{4b'-a'}{3} + a'\frac{4b'-a'}{3} + a'\frac{4b'-a'}{3}) \\
&\quad -\frac{(a'-b')^2}{9}\lambda(E_{(t-r+1)(t-r+2)}, \vec{y'}).
\end{aligned} \tag{24}$$

Again consider a new weighting for G_1: $\vec{y''} = (y''_1, y''_2, \dots, y''_t)$ given by $y''_i = z'_i$ for $i \ne t-r-1$, $i \ne t-r$ and $y''_{t-r-1} = z'_{t-r-1} - \frac{a'-b'}{3} = \frac{2a'+b'}{3}$, $y''_{t-r} = z'_{t-r} + \frac{a'-b'}{3} = \frac{2a'+b'}{3}$. Clearly, $\vec{y''} = (y''_1, y''_2, \dots, y''_t)$ is also

a legal weighting for G_1, and

$$\begin{aligned}\lambda(G_1,\vec{y''})-\lambda(G_1,\vec{z'}) &= \frac{a'-b'}{3}[\lambda(E_{t-r},\vec{z'})-\lambda(E_{t-r-1},\vec{z'})]-(\frac{a'-b'}{3})^2\lambda(E_{(t-r-1)(t-r)},\vec{z'})\\ &= \frac{a'-b'}{3}(a'-\frac{a'+2b'}{3})\lambda(E_{(t-r-1)(t-r)},\vec{z'})-(\frac{a'-b'}{3})^2\lambda(E_{(t-r-1)(t-r)},\vec{z'})\\ &= \frac{(a'-b')^2}{9}\lambda(E_{(t-r-1)(t-r)},\vec{z'}). \qquad (25)\end{aligned}$$

Consider a new weighting for G_1 once more: $\vec{z''}=(z''_1,z''_2,\ldots,z''_t)$ given by $z''_i=y''_i$ for $i\neq t-r-2$, $i\neq t-r+1$ and $z''_{t-r-2}=y''_{t-r-2}-\frac{a'-b'}{3}=\frac{2a'+b'}{3}$, $z''_{t-r+1}=y''_{t-r+1}+\frac{a'-b'}{3}=\frac{2a'+b'}{3}$. Clearly, $\vec{z''}=(z''_1,z''_2,\ldots,z''_t)$ is also a legal weighting for G_1, and

$$\begin{aligned}\lambda(G_1,\vec{z''})-\lambda(G_1,\vec{y''}) &= \frac{a'-b'}{3}[\lambda(E_{t-r+1},\vec{y''})-\lambda(E_{t-r-2},\vec{y''})]-(\frac{a'-b'}{3})^2\lambda(E_{(t-r-2)(t-r+1)},\vec{y''})\\ &= \frac{a'-b'}{3}(a'-\frac{a'+2b'}{3})\lambda(E_{(t-r-2)(t-r+1)},\vec{y''})-(\frac{a'-b'}{3})^2\lambda(E_{(t-r-2)(t-r+1)},\vec{y''})\\ &= \frac{(a'-b')^2}{9}\lambda(E_{(t-r-2)(t-r+1)},\vec{y''}). \qquad (26)\end{aligned}$$

Adding (21), (23), (24),(25),and (26), we have

$$\begin{aligned}\lambda(G_1,\vec{z''})-\lambda(G_3,\vec{x'}) &= \frac{(a'-b')^2}{9}[\lambda(E_{(t-r-1)(t-r)},\vec{z'})+\lambda(E_{(t-r-2)(t-r+1)},\vec{y''})-\\ &\quad \lambda(E_{(t-r)(t-r+3)},\vec{x'})-\lambda(E_{(t-r+1)(t-r+2)},\vec{y'})]+\frac{a'-b'}{3}c'^{r-3}(b'^2+2a'b')\\ &\quad -\frac{a'-b'}{3}c'^{r-3}(\frac{a'+2b'}{3}\frac{4b'-a'}{3}+a'\frac{4b'-a'}{3}+a'\frac{4b'-a'}{3})\\ &\quad +(2b^3-2a'b'^2)c'^{r-3}. \qquad (27)\end{aligned}$$

Note that

$$\begin{aligned}&\lambda(E_{(t-r-2)(t-r+1)},\vec{y''})-\lambda(E_{(t-r-2)(t-r+1)},\vec{z'})\\ =\ &\frac{a'-b'}{3}[\lambda(E_{(t-r-2)(t-r)(t-r+1)},\vec{z'})-\lambda(E_{(t-r-2)(t-r-1)(t-r+1)},\vec{z'})]\\ &-\frac{(a'-b')^2}{9}\lambda(E_{(t-r-2)(t-r-1)(t-r)(t-r+1)},\vec{z'})\\ =\ &\frac{a'-b'}{3}(z'_{t-r-1}-z'_{t-r})\lambda(E_{(t-r-2)(t-r-1)(t-r)(t-r+1)},\vec{z'})\\ &-\frac{(a'-b')^2}{9}\lambda(E_{(t-r-2)(t-r-1)(t-r)(t-r+1)},\vec{z'})\\ =\ &\frac{(a'-b')^2}{9}\lambda(E_{(t-r-2)(t-r-1)(t-r)(t-r+1)},\vec{z'})\geq 0, \qquad (28)\end{aligned}$$

also note that

$$\lambda(E_{(t-r-2)(t-r+1)},\vec{z'})\geq\lambda(E_{(t-r-2)(t-r+2)},\vec{z'}) \qquad (29)$$

since G_1 is left-compressed; and note that

$$\lambda(E_{(t-r-2)(t-r+2)},\vec{z'})-\lambda(E_{(t-r-2)(t-r+2)},\vec{y'})=\frac{a'-b'}{3}\lambda(E_{(t-r-2)(t-r+1)(t-r+2)},\vec{y'})\geq 0. \qquad (30)$$

Using (28), (29), and (30), we have

$$\lambda(E_{(t-r-2)(t-r+1)}, \vec{y''}) \geq \lambda(E_{(t-r-2)(t-r+2)}, \vec{y'}).$$

Therefore,

$$\begin{aligned}
& \lambda(E_{(t-r-1)(t-r)}, \vec{z'}) + \lambda(E_{(t-r-2)(t-r+1)}, \vec{y''}) \\
& -\lambda(E_{(t-r)(t-r+3)}, \vec{x'}) - \lambda(E_{(t-r+1)(t-r+2)}, \vec{y'}) \\
\geq\ & \lambda(E_{(t-r+1)(t-r+2)}, \vec{z'}) + \lambda(E_{(t-r-2)(t-r+2)}, \vec{y'}) \\
& -\lambda(E_{(t-r)(t-r+3)}, \vec{y'}) - \lambda(E_{(t-r+1)(t-r+2)}, \vec{y'}) \geq 0 \qquad (31)
\end{aligned}$$

since $\lambda(E_{(t-r+1)(t-r+2)}, \vec{z'}) = \lambda(E_{(t-r+1)(t-r+2)}, \vec{y'})$.

Recall that $a' \leq 2b'$. So

$$\begin{aligned}
& \frac{a'-b'}{3}c'^{r-3}(b'^2 + 2a'b') + \frac{a'-b'}{3}c'^{r-3}\left(\frac{a'+2b'}{3}\frac{4b'-a'}{3} + a'\frac{4b'-a'}{3} + a'\frac{4b'-a'}{3}\right) + \\
& +(2b'^3 - 2a'b'^2)c'^{r-3} \\
=\ & \frac{a'-b'}{3}c'^{r-3}\left(b'^2 + 2a'b' + \frac{a'+2b'}{3}\frac{4b'-a'}{3} + 2a'\frac{4b'-a'}{3} - 6b'^2\right) \\
=\ & \frac{a'-b'}{27}c'^{r-3}(44a'b' - 37b'^2 - 7a'^2) \\
=\ & \frac{(a'-b')^2}{27}c'^{r-3}(37b' - 7a') \geq 0. \qquad (32)
\end{aligned}$$

Combing (27), (31) and (32), we have

$$\lambda(G_1) \geq \lambda(G_1, \vec{z''}) \geq \lambda(G_3, \vec{x'}) = \lambda(G_3).$$

This proves the theorem. ■

Proof of Corollary 1.10. Let m and t be integers satisfying $\binom{t-1}{3} \leq m \leq \binom{t}{3} - 1$. Let $G = (V, E)$ be a 3-graph with m edges such that $\lambda(G) = \lambda^3_m$. Applying Lemma 1.6, we can assume that G is left-compressed. Let $\vec{x} = (x_1, x_2, \ldots, x_n)$ be an optimal weighting for G satisfying $x_1 \geq x_2 \geq \ldots \geq x_k > x_{k+1} = \ldots = x_n = 0$.

We claim that $k \leq t$. Otherwise $k \geq t+1$ and Lemma3.3 implies that

$$\begin{aligned}
m = |E| &\geq \binom{k-1}{3} + \binom{k-2}{2} - (k-2) \\
&\geq \binom{t}{3} + \binom{t-1}{2} - (t-1) \\
&\geq \binom{t}{3}
\end{aligned}$$

which contradicts to the assumption that $m = \binom{t}{3} - 4$. Hence $k \leq t$. Combining with Theorem 1.9, we see that the corollary holds. ■

Remark 4.1 *If a result similar to Lemma 3.3 holds for $r \geq 4$, then combining with Theorems 1.7 and 1.9, Conjecture 1.3 holds for general $r \geq 4$ when $m = \binom{t}{r} - 3$ or $m = \binom{t}{r} - 4$.*

References

[1] P. Frankl and Z. Füredi, Extremal problems whose solutions are the blow-ups of the small Witt-designs, *Journal of Combinatorial Theory (A)* 52 (1989), 129-147.

[2] P. Frankl and V. Rödl, Hypergraphs do not jump, *Combinatorica* 4 (1984), 149-159.

[3] P. Frankl and V. Rödl, Some Ramsey-Turán type results for hypergraphs, *Combinatorica* 8 (1989), 323-332.

[4] G. He, Y. Peng, and C. Zhao, On finding Lagrangians of 3-uniform hypergraphs, Ars Combinatoria (accepted).

[5] T.S. Motzkin and E.G. Straus, Maxima for graphs and a new proof of a theorem of Turán, *Canad. J. Math* 17 (1965), 533-540.

[6] D. Mubayi, A hypergraph extension of Turan's theorem, J. Combin. Theory Ser. B 96 (2006), 122-134.

[7] Y. Peng and C. Zhao, A Motzkin-Straus type result for 3-uniform hypergraphs, Graphs and Combinatorcs, published online 2012, DOI: 10.1007/s00373-012-1135-5.

[8] Y. Peng, Q. S. Tang, and C. Zhao, On Lagrangians of r-uniform Hypergraphs, submitted.

[9] S. Rota Buló, and M. Pelillo, A continuous characterization of maximal cliques in k-uniform hypergraphs. In Learning and Intellig. Optim. (Lecture Notes in Computer Science), Vol.5313 (2008), 220-233.

[10] S. Rota Buló and M. Pelillo, A generalization of the Motzkin-Straus theorem to hypergraphs. Optim. Letters 3, 2 (2009), 287-295.

[11] A. F. Sidorenko, Solution of a problem of Bollobas on 4-graphs, Mat. Zametki 41 (1987), 433-455.

[12] A. F. Sidorenko, Boundedness of optimal matrices in extremal multigraph and digraph problems, Combinatorica, 13(1), 1993, 109-120.

[13] V. Sós, and E. G. Straus, Extremal of functions on graphs with applications to graphs and hypergraphs. J. Combin. Theory Series B 63 (1982), 189-207.

[14] J. Talbot, Lagrangians of hypergraphs, *Combinatorics, Probability & Computing* 11 (2002), 199-216.

[15] P. Turán, On an extremal problem in graph theory(in Hungarian), *Mat. Fiz. Lapok* 48 (1941), 436-452.